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THE AXISYMMETRIC PROBLEM OF TRANSFERRING A LOAD TO A VISCOELASTIC ORTHOTROPIC BODY BY MEANS OF AN ELASTIC ROD*

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The problem of transferring a load by means of an elastic rod to a viscoelastic orthotropic body possessing cylindrical anisotropy, is studied. An asymptotic method /1, 2/ is used, which is generalized in the first part of this paper to the case of viscoelastic media.

When the problem of the adhesive strength of a fibrous composite material is solved, an important role is played by the problems of transferring the load to three-dimensional bodies by means of reinforcing elements. Such problems have not been studied sufficiently even in the elastic domain, due to considerable mathematical difficulties /1, 3-5/.

1. Let us consider a viscoelastic orthotropic body with cylindrical anisotropy. When the load is axially symmetric, the stress tensor and displacement vector do not depend on θ and are functions of the coordinates r, z (r, θ, z are cylindrical coordinates and the z axis coincides with the axis of anisotropy). In this case the problem splits into two independent problems. The problem of deformation in which there is no component of the displacement v (but we have, of course, the normal stress σ_{22}), and the problem of torsion. Let us consider the first problem.

The relations connecting the deformation and the stresses in an orthotropic viscoelastic body with cylindrical anisotropy in which there is no component of the displacement v, are written as follows:

$$e_{11} = s_1 - \nu_{12}s_2 - \nu_{13}s_3 \tag{1.1}$$
$$s_i = \frac{1}{E_i}\left(\sigma_{ii} + \int_0^t K_{1i}(t-\tau)\,\sigma_{ii}\,d\tau\right), \quad i = 1, 2, 3$$
$$e_{13} = \frac{1}{G}\left(\sigma_{13} + \int_0^t K(t-\tau)\,\sigma_{13}\,d\tau\right), \quad e_{12} = e_{23} = 0$$

To obtain e_{22}, e_{33}, it is sufficient to carry out a cyclic permutation of the indices in e_{11}, and here we have

$$\nu_{12}E_1 = \nu_{21}E_2, \quad \nu_{23}E_2 = \nu_{32}E_3, \quad \nu_{31}E_3 = \nu_{13}E_1$$
$$K_{12} = K_{21}, \quad K_{23} = K_{32}, \quad K_{31} = K_{13}$$

Here E_1, E_2, E_3 (G) are the instantaneous elasticity (shear) moduli, ν_{ij} are Poisson's ratios, $\sigma_{11}, \sigma_{22}, \sigma_{33}$ $(\sigma_{13} = \sigma_{31})$ are the normal (tangential) stresses, $K_{ij}(t-\tau)$ are the creep kernels and t is the time. Relations (1.1) hold in the case of tension, as well as compression. We use the following analytic expressions /6/ to approximate the creep kernels:

$$K_{ij}(t-\tau) = k_{ij}(t-\tau)^{\alpha_{ij}-1}\exp[-\beta_{ij}(t-\tau)] \tag{1.2}$$
$$K(t-\tau) = k(t-\tau)^{\alpha-1}\exp[-\beta(t-\tau)] \quad (0 < \alpha_{ij}, \alpha \leqslant 1)$$

The deformation tensor components are expressed in terms of the projections u, w of the displacement vector, according to the formulas

$$e_{11} = \frac{\partial u}{\partial r}, \quad e_{22} = \frac{u}{r}, \quad e_{33} = \frac{\partial w}{\partial z}, \quad e_{13} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \tag{1.3}$$

After applying Laplace transformations to relations (1.1) and (1.3) and constructing the differential equations in the usual manner, we arrive at the problem of integrating a system of equations analogous to (1.1) of /1/, where the parameter ε has the form ($\Gamma(\alpha)$ is the

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Gamma function)

$$\varepsilon = \varepsilon_1 F(p; k, \alpha, \beta)/F(p; k_{11}, \alpha_{11}, \beta_{11}) \tag{1.4}$$
$$\varepsilon_1 = G/E_1, \quad F(p; k, \alpha, \beta) = [1 + k\Gamma(\alpha)(p+\beta)^{-\alpha}]^{-1}$$

In carrying out the asymptotic analysis of the resulting equation we use, as in /1/, the small parameter ε. The parameter will indeed be small if ε_1 is small, since the coefficient accompanying ε_1 in expression (1.4) is of the order of unity for any value of the parameter p.

It follows therefore, that, if we use the difference creep kernels (1.2), then we can split the stress-deformation state into two components, each with different properties, also in the case of axisymmetric problems of linear viscoelasticty of orthotropic media. The complete solution of the problem will be sought in the form of a super-position of both components, and the determination of each component will be reduced to solving the boundary-value problems for a single function.

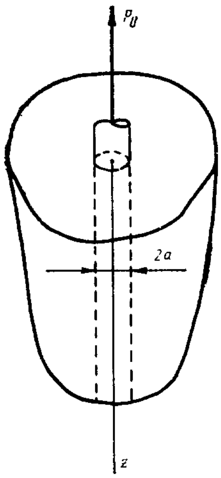

Fig.1

2. Let us now consider the problem of transferring an axial load, by means of an elastic rod, to a viscoelastic orthotropic body with cylindrical anisotropy. The rod is inserted into the body so that it is perpendicular to the boundary plane, and its middle line coincides with the z axis (Fig.1). The area F of transverse cross-section of the rod is fairly small, i.e. its radius a is fairly small. We require to determine the law of distribution of contact stresses between the rod and the half-space, when a concentrated force P_0 is applied to the end point of the rod $z = 0$ at the initial instant of time, directed along the rod axis and remaining constant from then on.

The problem in question is solved in mappings as in /1/, and the displacement transform will have the form

$$W(r, z, p) = -\frac{2P_0}{\pi EF}\frac{1}{p}\int \frac{K_0(r\omega s)\cos zs}{s[sK_0(a\omega s) + gK_1(a\omega s)]}\,ds$$
$$g = 2\pi a G F(p; k, \alpha, \beta)\,\omega/(EF)$$

Here $K_0(x), K_1(x)$ are modified Bessel functions, and the integration in s is carried out everywhere from 0 to ∞. The transforms of the force within the rod $N_* = EF(dW/dz)|_{r=a}$ and of contact stress $\tau_* = 2\pi a G F(p; k, \alpha, \beta)(dW/dr)|_{r=a}$ are given by the formulas

$$N_*(z, p) = \frac{2P_0}{\pi}\frac{1}{p}\int \frac{\sin zs}{s + gM(s)}\,ds \tag{2.1}$$
$$\tau_*(z, p) = \frac{2P_0}{\pi}\frac{g}{p}\int \frac{\cos zs}{g + sM^{-1}(s)}\,ds$$
$$M(s) = K_1(a\omega s)/K_0(a\omega s)$$

Using an inverse Laplace transformation we can successfully reduce the contour integral to an ordinary, non-singular integral, using the method given in /6/. The asymptotic expressions for the force shown for small and large values of time, can be written in explicit form provided that their transforms can be written in the form of a series in the small parameter ε_* depending on p

$$T_*(z, p) = [T_0(z) + T_1(z)\varepsilon_* + T_2(z)\varepsilon_*^2 + \ldots]/p \tag{2.2}$$

where T is understood to represent either the force N, or τ.

If the material of the half-space has predominant shear creep $(k_{ij} = 0)$ and $\alpha = 1$, then

$$\omega = \omega_0[(p+\beta+k)/(p+\beta)]^{1/2}, \quad g = g_0[(p+\beta)/(p+\beta+k)]^{1/2}$$
$$\omega_0 = (E_3/G)^{1/2}, \quad g_0 = 2\pi a(E_3G)^{1/2}/(EF)$$

In this case we have in series (2.2) for large values of the parameter p (which corresponds to small values of the time t) $\varepsilon_* = k/(p+\beta)$, and for small values (corresponding to large time values), $\varepsilon_* = \Delta p/(p+\beta)$, $\Delta = -k/(\beta+k)$.

Passing to the original in (2.2) we obtain, for small values of time,

$$T(z, t) = T_0(z) + T_{10}(z)(k/\beta)(1 - e^{-\beta t}) + \ldots \tag{2.3}$$

where we have, for the force $N(z, t)$ within the rod,

$$T_0(z) = \frac{2P_0}{\pi}\int \frac{\sin zs}{s + g_0M_0(s)}\,ds \tag{2.4}$$

$$T_{10}(z) = \frac{2P_0}{\pi}\int \frac{[g_0 a\omega_0 s[1 - M_0^2(s)] + 2g_0 M_0(s)}{2[s + g_0 M_0(s)]^2}\sin zs\,ds$$

and for the contact stress $\tau(z, t)$ we have

$$T_0(z) = \frac{2P_0 g_0}{\pi}\int \frac{\cos zs}{g_0 + sM_0^{-1}(s)}ds \tag{2.5}$$

$$T_{10}(z) = \frac{2P_0 g_0}{\pi}\int \frac{a\omega_0 s^2[1 - M_0^{-2}(s) - 2sM_0^{-1}(s)]}{2[g_0 + sM_0^{-1}(s)]^2}\cos zs\,ds$$

$$M_0(s) = K_1(a\omega_0 s)/K_0(a\omega_0 s)$$

The coefficients $T_{20}(z), \ldots$ are not given here because of their bulk.

At large values of the time, (2.2) gives

$$T(z, t) = T_\infty(z) + T_{1\infty}(z)\Delta e^{-\beta t} + \ldots \tag{2.6}$$

Here $T_\infty(z), T_{1\infty}(z), \ldots$ are found from the same formulas (2.4) and (2.5) after replacing ω_0 by ω_∞, g_0 by g_∞, and $\omega_\infty = \omega_0(1 + k/\beta)^{1/2}$, $g_\infty = g_0(1 + k/\beta)^{-1/2}$.

The forces (2.3) and (2.6) can be represented by asymptotic expressions for small and large values of the coordinate z, analogous to the elastic problem in /1/. In particular, when $t = 0$ and the coordinate z is small, we obtain

$$N_0(z) = \frac{2P_0}{\pi}(\sin z_1 \operatorname{ci} z_1 - \cos z_1 \operatorname{si} z_1), \quad z_1 = g_0 z \tag{2.7}$$

$$\tau_0(z) = -\frac{2P_0 g_0}{\pi}(\cos z_1 \operatorname{ci} z_1 + \sin z_1 \operatorname{si} z_1)$$

The limit forces (when $t = 0$) for small values of the coordinate z are also given by the formulas (2.7), but g_0 must be replaced by g_∞. The manner in which the forces decrease at large values of z, is analogous to that in the elastic problem /1/.

3. Relations (2.3) and (2.6) can be written in the form

$$T(t) = a_0 + a_1 t + \ldots (t \to 0),\ T(t) = b_0 + b_1 e^{-\beta t} + \ldots (t \to \infty) \tag{3.1}$$

The limiting information supplied by (3.1) enables us to make a judgement about the behaviour of the corresponding functions at arbitrary times, provided that we use the two-point Padé approximant /7/. This approximant is a function of the form

$$T(t) = \frac{\alpha_0 + \alpha_1 t + \beta_1 e^{\beta t} + \ldots}{1 + \gamma_1 t + \delta_1 e^{\beta t} + \ldots} \tag{3.2}$$

whose coefficients are chosen from the conditions that expansion in series as $t \to 0$ and $t \to \infty$ yields the asymptotic expressions (3.1). These conditions yield

$$\alpha_0 = b_0 + b_1\delta_1,\quad \alpha_1 = b_0\gamma_1,\quad \beta_1 = b_0\delta_1$$

$$\delta_1 = -\frac{b_0 - a_0}{b_0 + b_1 - a_0},$$

$$\gamma_1 = \frac{a_1 b_1 + \beta(b_0 - a_0)^2}{(b_0 - a_0)(b_0 + b_1 - a_0)}$$

The Padé approximant (3.2) indeed enables us to "match" the limit expressions (3.1) and to find the regions of "small" and "large" values of time.

Fig.2 shows the results of calculating, according to formulas (2.7), the conditions within the rod $N^* = N/P_0$ and the contact forces $\tau^* = \tau/(P_0 g_0)$ and $t = 0$ (curves *1* and *2* respectively), and at $t = \infty$ (curves *3* and *4*). The insert in the upper right corner shows the variation in the force N^* with time when $z_1 = 0.2$. The dashed curves *1* and *2* were constructed using the formulas (3.1), and the dot-dash curve was obtained using the formula (3.2). The computations were carried out for $k = 2.5$, $\beta = 0.5$.

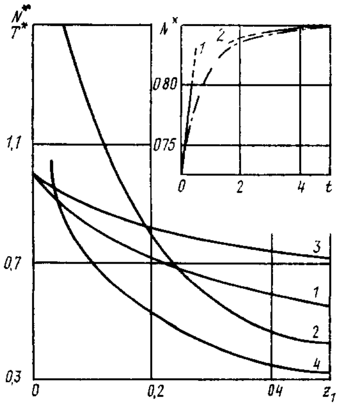

Fig.2

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THE COMBINED PROBLEM OF THERMOELASTIC CONTACT BETWEEN TWO PLATES THROUGH A HEAT CONDUCTING LAYER*

V.V. ZOZULYA

The problem of the contact between two plates under the action of a force and temperature field is formulated. It is assumed that when the plates are deformed, the conditions of heat exchange between them also change. The equations of motion and heat conduction of the thermoelastic plates, as well as the equations of heat conduction of the heat conducting layer are derived by expanding the three-dimensional equations in series in Legendre polynomials. The equations of the n-th approximation are constructed and the equations of the first approximation are studied in detail.

1. Formulation of the problem. We consider two plates (*1* and *2*) of arbitrary contour and constant thickness h_1 and h_2, respectively, situated, in an initial undeformed state, a distance h_0 apart. We shall assume that h_0 is commensurable with the flexures of the plates, and we will assume the flexures to be small. A heat-conducting medium is enclosed between the plates. The medium does not resist their deformation, and heat exchange within it is due to its thermal conductivity. Let $\Omega_\gamma\ (\gamma = 1, 2)$ be the regions occupied by the median surfaces of the plates, $\partial\Omega_\gamma$ their boundaries, Ω_γ^+ and Ω_γ^- the upper and lower surfaces of the plates, and Γ_γ the side surfaces.

The thermodynamic state of the system, including the plates and heat conducting layer, is defined by the following parameters: $\sigma_{ij(\gamma)}(x_k, t)$, $\varepsilon_{ij(\gamma)}(x_k, t)$, $u_{i(\gamma)}(x_k, t)$ $(i, j, k = 1, 2, 3;\ \gamma = 1, 2)$ are the components of the stress and deformation tensors and displacement vectors of the plates, and $T_\gamma(x_k, t)$, $\chi_\gamma(x_k, t)$, $T_*(x_k, t)$, $\chi_*(x_k, t)$ are the temperature and specific strength of the internal heat sources in the plates and the layer, respectively. The boundary conditions written in terms of the stresses and conditions of heat exchange with external medium and with the heat conducting layer are specified at the end surfaces Ω_γ^+ and Ω_γ^-. The boundary conditions at the sides consist of mechanical and thermal conditions and depend on the way they are clamped and on the heat exchange conditions. The distribution of the displacements, velocities and temperature in the plates and the layer at the initial instant $t = 0$, are known.

The external forces and temperature fields acting on the plates cause them to bend towards each other, and the plates may come into contact. This is accompanied by the appearance of a previously unknown zone of dense contact $\Omega_e(t) = \Omega_1^- \cap \Omega_2^+$ changing with time, within which the contact forces of interaction $q_i(x_\alpha, t)$ $(i = 1, 2, 3;\ \alpha = 1, 2)$ appear and contact heat transfer occurs. The problem therefore consists of determining the stress-deformation state and the temperature fields within the plates, the region of dense (complete) contact

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